

An equivariant index for proper actions III: the invariant and discrete series indices

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Abstract

We study two special cases of the equivariant index defined in part I of this series. We apply this index to deformations of Spin^c -Dirac operators, invariant under actions by possibly noncompact groups, with possibly noncompact orbit spaces. One special case is an index defined in terms of multiplicities of discrete series representations of semisimple groups, where we assume the Riemannian metric to have a certain product form. The other is an index defined in terms of sections invariant under a group action. We obtain a relation with the analytic assembly map, quantisation commutes with reduction results, and Atiyah–Hirzebruch type vanishing theorems. The arguments are based on an explicit decomposition of Spin^c -Dirac operators with respect to a global slice for the action.

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1 Introduction

In part I of this series [17], an equivariant index was defined for actions by possibly noncompact groups, with possibly noncompact orbit spaces. It was shown to apply to natural deformations of Dirac-type operators as in [7, 8, 18, 19], which have been used successfully in geometric quantisation [14, 17, 29].

In this paper, we consider such deformations of Spin^c -Dirac operators. For actions by semisimple Lie groups with discrete series, on manifolds with Riemannian metrics of a certain product form, the equivariant index can be expressed in terms of multiplicities of discrete series representations in the kernel of such an operator. This motivates the definition of the *discrete series index*. We also show that, for these deformed Spin^c -Dirac operators, the equivariant index generalises the *invariant index* studied in [8, 13]. The latter is defined in terms of sections invariant under a group action.

The assumption on the Riemannian metric means direct geometric arguments can be used to obtain these relations with the equivariant index. (For the invariant index, this assumption is not necessary because the claim can always be reduced to metrics of that form.) In the cocompact case, the discrete series index is also directly related to the analytic assembly map [5, 20]. Furthermore, we obtain *quantisation commutes with reduction* results

for the discrete series index and the invariant index. The result for the invariant index sharpens an asymptotic result in [14]. In the cocompact case, this result reduces to a Spin^c -version of Landsman's conjecture [16, 22]. Finally, Atiyah and Hirzebruch's vanishing result [3] on compact Spin manifolds generalises to the discrete series index and the invariant index in a way analogous to the K-theoretic result in [15]. (The result for the invariant index is actually a special case of the result in [15].)

As said above, the results for the discrete series index hold for Riemannian metrics of a certain form. It is an interesting question to what extent they generalise to arbitrary (complete, invariant) metrics, as is the case for the invariant index.

Overview

In Section 2, we recall the definitions of the equivariant index of [17] and the invariant index of [8, 13], introduce the discrete series index, and state the main results. In Section 3 we give a decomposition of Spin^c -Dirac operators in terms of a global slice for the action. This leads to an induction result, Proposition 3.1. Then, in Section 4, we include L^2 -inner products to decompose the kernels of Dirac operators on the relevant spaces. This allows us to prove the main results in Subsection 4.4.

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2 Preliminaries and results

Throughout this paper, we consider a complete Riemannian manifold M . We will identify $T^*M \cong TM$ via the Riemannian metric where convenient. Furthermore, G will be a Lie group, with a maximal compact subgroup K . Unless stated otherwise, we assume that G is unimodular, with a bi-invariant Haar measure dg , and that G/K is even-dimensional. (Some of the constructions in this paper apply to more general groups, but we only apply them under these assumptions here.) We suppose G acts properly and isometrically on M . Let $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^- \rightarrow M$ be a \mathbb{Z}_2 -graded, G -equivariant Hermitian vector bundle. In most of this paper, \mathcal{S} will be the spinor bundle of an equivariant Spin^c -structure. For any (odd) operator D on sections of \mathcal{S} , we write D^\pm for the restriction of D to sections of \mathcal{S}^\pm .

The proofs of the results stated in this section are given in Subsection 4.4.

2.1 Product metrics

The results about the discrete series index in this paper hold for Riemannian metrics on TM of a certain form. By Abels' theorem [1], there is a K -invariant submanifold $N \subset M$ such that the action map defines a G -equivariant diffeomorphism

$$G \times_K N \cong M. \quad (2.1)$$

Here the left hand side is the quotient of $G \times N$ by the action by K given by $k \cdot (g, n) = (gk^{-1}, kn)$, for $k \in K$, $g \in G$ and $n \in N$.

Let $\mathfrak{p} \subset \mathfrak{g}$ be a K -invariant subspace such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then under (2.1), we have

$$TM = G \times_K (TN \oplus (N \times \mathfrak{p})) \rightarrow M. \quad (2.2)$$

Definition 2.1. A *product metric* on TM is a G -invariant Riemannian metric induced by a K -invariant Riemannian metric on TN and a K -invariant inner product on \mathfrak{p} via the isomorphism (2.2).

In the results about discrete series representations, the Riemannian metric on TM will be assumed to be a product metric. This will be indicated in the relevant places.

2.2 A realisation of discrete series representations

The explicit realisation of discrete series representations by Atiyah and Schmid [4] and Parthasarathy [28] plays an important role in this paper. This realisation involves Dirac operators on G/K .

Let G be any Lie group. Fix a K -invariant inner product on \mathfrak{p} . Let $\pi_{\mathfrak{p}}$ be the standard representation of $\text{Spin}(\mathfrak{p})$. Since G/K is even-dimensional, $\pi_{\mathfrak{p}}$ splits as $\pi_{\mathfrak{p}} = \pi_{\mathfrak{p}}^+ \oplus \pi_{\mathfrak{p}}^-$. Let $D_{G/K}$ be the operator

$$D_{G/K} = \sum_{j=1}^k X_j \otimes c(X_j) \quad (2.3)$$

on $C^\infty(G) \otimes \pi_{\mathfrak{p}}$, where $\{X_1, \dots, X_k\}$ is an orthonormal basis of \mathfrak{p} . Here $c: \mathfrak{p} \rightarrow \text{End}(\pi_{\mathfrak{p}})$ is the Clifford action.

Throughout this paper, we assume that the adjoint representation

$$\text{Ad}: K \rightarrow \text{SO}(\mathfrak{p})$$

lifts to $\text{Spin}(\mathfrak{p})$, i.e. that G/K is G -equivariantly Spin. This is true for a double cover of G . Via this lift, we view $\pi_{\mathfrak{p}}$ as a representation of K . We will write $\pi_{\mathfrak{p}}$ for the formal difference

$$\pi_{\mathfrak{p}} := \pi_{\mathfrak{p}}^+ - \pi_{\mathfrak{p}}^- \in \mathcal{R}(K).$$

Consider the diagonal representation of K in $C^\infty(G) \otimes \pi_{\mathfrak{p}}$. The space $(C^\infty(G) \otimes \pi_{\mathfrak{p}})^K$ is the space of smooth sections of the spinor bundle $G \times_K \pi_{\mathfrak{p}} \rightarrow G/K$.

Now suppose G is connected and semisimple with discrete series, i.e. $\text{rank}(G) = \text{rank}(K)$. Then by Proposition 1.1 in [28], the restriction of $D_{G/K}$ to $(C^\infty(G) \otimes \pi_{\mathfrak{p}})^K$ is the Spin-Dirac operator on G/K . Let $T < K$ be a maximal torus, with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$. Let $\mathfrak{t}_+^* \subset \mathfrak{t}^*$ be a choice of (closed) positive Weyl chamber. Let R be the set of roots of $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and let R^+ be the set of positive roots with respect to \mathfrak{t}_+^* . We denote half the sum of these positive roots by ρ_K . Let $\Lambda_+ \subset \mathfrak{t}^*$ be the set of dominant integral weights with respect to \mathfrak{t}_+^* . In the Spin^c -setting, it is natural to parametrise the irreducible representations of K by their infinitesimal characters, rather than by their highest weights. For $\lambda \in \Lambda_+ + \rho_K$, let π_λ^K be the irreducible representation of K with infinitesimal character λ , i.e. with highest weight $\lambda - \rho_K$.

The discrete series of G was realised in Theorem 9.3 in [4] and Theorem 1 in [28].

Theorem 2.2 (Atiyah–Schmid, Parthasarathy). *Let $\lambda \in \Lambda_+ + \rho_K$. One has*

$$(\pi_\lambda^K \otimes \ker_{L^2}(D_{G/K}^-))^K = 0.$$

If λ is singular, then also

$$(\pi_\lambda^K \otimes \ker_{L^2}(D_{G/K}^+))^K = 0.$$

If λ is regular, then

$$(\pi_\lambda^K \otimes \ker_{L^2}(D_{G/K}^+))^K = \pi_\lambda^{\text{ds}},$$

where π_λ^{ds} is the discrete series representation of G with Harish–Chandra parameter λ .

2.3 Spin^c -Dirac operators and the equivariant index

We return to case where G is any Lie group. Now suppose that M is even-dimensional, and that it has a G -equivariant Spin^c -structure. The vector bundle \mathcal{S} is taken to be the spinor bundle associated to the Spin^c -structure. We denote the determinant line bundle of the Spin^c -structure by $L \rightarrow M$, and choose a G -invariant Hermitian connection ∇^L on L . Together with the Levi-Civita connection on TM , this induces a connection $\nabla^{\mathcal{S}}$ on \mathcal{S} (see e.g. Proposition D.11 in [23]). This in turn defines a Spin^c -Dirac operator D on \mathcal{S} , by

$$D: \Gamma^\infty(\mathcal{S}) \xrightarrow{\nabla^{\mathcal{S}}} \Omega^1(M; \mathcal{S}) \xrightarrow{c} \Gamma^\infty(\mathcal{S}).$$

Here c denotes the Clifford action by $TM \cong T^*M$ on \mathcal{S} .

Let

$$\mu: M \rightarrow \mathfrak{g}^*$$

be the Spin^c -momentum map, defined by

$$2i\mu_X = \mathcal{L}_X^L - \nabla_{X^M}^L \in \text{End}(L) = C^\infty(M, \mathbb{C}). \quad (2.4)$$

Here μ_X is the pairing of μ with an element $X \in \mathfrak{g}$, \mathcal{L}^L denotes the Lie derivative of sections of L , and X^M is the vector field induced by X . By Lemma 5.3 in [14], the connection ∇^L can be chosen so that $\mu(N) \subset \mathfrak{k}^*$. (We identify \mathfrak{k}^* with the annihilator of \mathfrak{p} in \mathfrak{g}^* .) We assume ∇^L was chosen in this way.

In addition, fix a K -invariant inner product $(-, -)^{\mathfrak{g}}$ on \mathfrak{g} extending the one on \mathfrak{p} , such that $\mathfrak{k} \perp \mathfrak{p}$. Consider the metric $\{(-, -)_m\}_{m \in M}$ on the trivial vector bundle $M \times \mathfrak{g} \rightarrow M$ defined by

$$(X, Y)_{gn} := (\text{Ad}(g)^{-1}X, \text{Ad}(g)^{-1}Y)^{\mathfrak{g}}$$

for $X, Y \in \mathfrak{g}$, $g \in G$ and $n \in N$. Let $\mu^*: M \rightarrow \mathfrak{g}$ be the map defined by

$$\langle \mu(m), X \rangle = (X, \mu^*(m))_m,$$

for $X \in \mathfrak{g}$ and $m \in M$. Consider the G -invariant vector field v on M defined by

$$v_m = 2(\mu^*(m))_m^M, \quad (2.5)$$

where $m \in M$, $(\mu^*(m))_m^M$ is the vector field induced by $\mu^*(m) \in \mathfrak{g}$, and the factor 2 was included for consistency with [13, 14, 29]. For a real-valued function $f \in C^\infty(M)^G$, the *Dirac operator deformed by fv* is the operator

$$D_{fv} := D + ic(fv) \quad (2.6)$$

on smooth sections of \mathcal{S} . Already in the compact case, such a deformation was used by Tian and Zhang [29] to prove Guillemin and Sternberg's quantisation commutes with reduction conjecture.

Assumption 2.3. *We assume that the zeroes of v form a cocompact subset of M .*

In [18], an equivariant index was defined for proper actions by possibly noncompact groups, with possibly noncompact orbit spaces. It was shown that this index applies to deformed Dirac operators (of a more general kind than the ones studied here). For any nonnegative function $\psi \in C^\infty(M)^G$, a nonnegative function $f \in C^\infty(M)^G$ is called *ψ -admissible* if, outside a cocompact subset of M , we have

$$\frac{f}{\|df\| + f + 1} \geq \psi.$$

By Theorem 3.12 in [18], there is a nonnegative function $\psi \in C^\infty(M)^G$ such that for all ψ -admissible functions $f \in C^\infty(M)^G$, we have a well-defined equivariant index

$$\text{index}_G(D_{fv}) := \left[L^2(\mathcal{S}), \frac{D_{fv}}{\sqrt{D_{fv}^2 + 1}}, \pi_{G, G/K} \right] \in \text{KK}(C_0(G/K) \rtimes G, \mathbb{C}). \quad (2.7)$$

Here $C_0(G/K) \rtimes G$ is a crossed-product C^* -algebra [30], and the $*$ -representation $\pi_{G, G/K}: C_0(G/K) \rtimes G \rightarrow \mathcal{B}(L^2(\mathcal{S}))$ is given by

$$(\pi_{G, G/K}(\varphi)s)(gn) = \int_G \varphi(g', gK) g' \cdot (s(g'^{-1}gn)) dg,$$

for $\varphi \in C_c(G, C_0(G/K))$, $s \in L^2(\mathcal{S})$, $g \in G$ and $n \in N$. Via the Morita equivalence $C_0(G/K) \rtimes G \sim C^*K$, this index can be identified with an element of $\text{KK}(C^*K, \mathbb{C}) = \hat{R}(K)$.

In the special case where $G = K$ is compact, the index (2.7) reduces to the index

$$\text{index}_K(D_{fv}) \in \hat{R}(K) \quad (2.8)$$

studied by Braverman in [7].

2.4 The invariant index

In [8, 13, 14], an index for proper actions is studied, defined with respect to sections invariant under the group action.

A section of a vector bundle invariant under a proper action by a non-compact group cannot be square-integrable. For that reason, we use a different Hilbert space of invariant sections. Let $h \in C^\infty(M)$ be a *cutoff function*, which means that it has compact support on G -orbits, and satisfies

$$\int_G h(gm)^2 dg = 1$$

for all $m \in M$. Then a section s of \mathcal{S} is called *transversally L^2* if hs is L^2 . This condition is independent of the cutoff function h if s is G -invariant. It was shown (for more general Dirac-type operators) in [8, 13] that there is a nonnegative function $\psi \in C^\infty(M)^G$ such that for all ψ -admissible $f \in C^\infty(M)^G$, the spaces

$$\ker_{L^2_T}(D_{fv}^\pm)^G := \{s \in \Gamma^\infty(\mathcal{S}^\pm)^G; s \text{ transversally } L^2 \text{ and } D_{fv}s = 0\}$$

are finite-dimensional. For such f , the *invariant index*

$$\text{index}_{L^2_T}^G(D_{fv}) := \dim(\ker_{L^2_T}(D_{fv}^+)^G) - \dim(\ker_{L^2_T}(D_{fv}^-)^G) \quad (2.9)$$

is independent of f . It is also independent of the Riemannian metric, as long as M is complete.

It was conjectured in Remark 4.4 in [19] that, for more general Dirac-type operators, the invariant index can be recovered from the equivariant index of (2.7) as in the following result for Spin^c -Dirac operators.

Proposition 2.4. *If f is ψ -admissible for ψ as in the definitions of the indices (2.7) and (2.9), then for any G -invariant, complete Riemannian metric on M ,*

$$\text{index}_{L^2_T}^G(D_{fv}) = \dim(\text{index}_G(D_{fv})^K),$$

where on the right hand side, we view $\text{index}_G(D_{fv})$ as an element of $\hat{R}(K)$.

2.5 The discrete series index

For now, let \mathcal{S} be any \mathbb{Z}_2 -graded, Hermitian, G -equivariant vector bundle, and let D be any odd, self-adjoint, G -equivariant operator on $L^2(\mathcal{S})$. We assume that G is connected and semisimple with discrete series. Let $\hat{G}_{ds} \subset \hat{G}$ be the discrete part of the unitary dual of G .

Definition 2.5. The *discrete series representation group* of G is the Abelian group

$$R_{ds}(G) := \left\{ \bigoplus_{\pi \in \hat{G}_{ds}} m_\pi \pi; m_\pi \in \mathbb{Z}, \text{ nonzero for finitely many } \pi \right\}.$$

The *completed discrete series representation group* of G is the Abelian group

$$\hat{R}_{ds}(G) := \left\{ \bigoplus_{\pi \in \hat{G}_{ds}} m_{\pi} \pi; m_{\pi} \in \mathbb{Z} \right\} \cong \text{Hom}_{\mathbb{Z}}(R_{ds}(G), \mathbb{Z}).$$

If $G = K$ is compact then we have $R_{ds}(K) = R(K)$ and $\hat{R}_{ds}(K) = \hat{R}(K)$, the usual representation ring and its completion.

Definition 2.6. If the multiplicity $[\ker_{L^2}(D) : \pi]$ of any $\pi \in \hat{G}_{ds}$ in the kernel of D is finite, then D is called *ds-Fredholm*, and its *discrete series index* is

$$\text{index}_{ds}(D) := \bigoplus_{\pi \in \hat{G}_{ds}} ([\ker_{L^2}(D^+) : \pi] - [\ker_{L^2}(D^-) : \pi]) \pi \in \hat{R}_{ds}(G).$$

Example 2.7. If $M = G/H$, for a compact subgroup $H < G$, and \mathcal{S} is a vector bundle associated to a finite-dimensional representation of H , then Theorem 6.1 in [9] states that any elliptic pseudo-differential operator D is ds-Fredholm. In fact, one has

$$\ker_{L^2}(D) \in R_{ds}(G)$$

for such operators. This was generalised to a larger class of groups in Theorem 6.2 in [9]. See also Proposition 7.3.A. in the same paper, for Dirac operators.

From now on, D will be a Spin^c -Dirac operator as in Subsection 2.3. For a real-valued function $f \in C^\infty(M)^G$, let D_{fv} be the deformed Spin^c -Dirac operator as in (2.6).

Proposition 2.8. *Suppose the Riemannian metric on TM is a product metric. Then there is a nonnegative function $\psi \in C^\infty(M)^G$ such that for all ψ -admissible functions f , the operator D_{fv} is ds-Fredholm. Its ds-index is independent of f and ∇^L . Furthermore, the L^2 -kernel of D_{fv} decomposes completely into discrete series representations.*

2.6 The discrete series index and other indices

Let G be connected and semisimple with discrete series. Consider the *Dirac induction map*

$$\widehat{D\text{-Ind}}_K^G : \hat{R}(K) \rightarrow \hat{R}_{ds}(G) \quad (2.10)$$

given by

$$\widehat{D\text{-Ind}}_K^G(\pi_\lambda^K) = (\pi_\lambda^K \otimes \ker_{L^2}(D_{G/K}^+))^K - (\pi_\lambda^K \otimes \ker_{L^2}(D_{G/K}^-))^K. \quad (2.11)$$

By Theorem 2.2, this map indeed takes values in $\hat{R}_{\text{ds}}(G)$, it is surjective, and its restriction to the part of $\hat{R}(K)$ spanned by representations with regular infinitesimal characters is an isomorphism of Abelian groups. Also, the second term on the right hand side of (2.11) is zero, but it was included for symmetry purposes.

In this subsection and the next, we suppose that the Riemannian metric on TM is a product metric.

Proposition 2.9. *Suppose $f \in C^\infty(M)^G$ is ψ -admissible for ψ both as in Proposition 2.8 and as in the definition of the index (2.7). Then*

$$\text{index}_G(D_{f_V}) = \pi_p \otimes (\widehat{D\text{-Ind}}_K^G)^{-1}(\text{index}_{\text{ds}}(D_{f_V})) \in \hat{R}(K).$$

Since G has discrete series representations, tensoring with π_p is an invertible operation (see Lemma 4.7 in [19]). So the equivariant index (2.7) determines the discrete series index of D_{f_V} .

Next, suppose that M/G is compact. Then we have the *analytic assembly map* [20] from the Baum–Connes conjecture [5]

$$\mu_M^G: K_0^G(M) \rightarrow K_0(C_r^*G).$$

Here $K_0^G(M)$ is the (even) G -equivariant K -homology of M , and $K_0(C_r^*G)$ is the (even) K -theory of the *reduced group C^* -algebra* of G . Consider the inclusion map

$$j: R_{\text{ds}}(G) \hookrightarrow K_0(C_r^*G)$$

given by $j(\pi) = [d_\pi c_\pi]$, where d_π is the formal degree of $\pi \in \hat{G}_{\text{ds}}$, and c_π is the matrix coefficient of any unit vector in the representation space of π (see [21]). Let $p_{\text{ds}}: K_0(C_r^*G) \rightarrow K_0(C_r^*G)$ be the projection onto the image of j .

Proposition 2.10. *If M/G is compact, then the Spin^c -Dirac operator D is ds -Fredholm, its ds -index lies in $R_{\text{ds}}(G)$, and we have*

$$j(\text{index}_{\text{ds}}(D)) = (-1)^{\dim(G/K)/2} p_{\text{ds}}(\mu_M^G[D]).$$

2.7 Spin^c -quantisation commutes with reduction for the discrete series

The *quantisation commutes with reduction* principle of Guillemin and Sternberg [10] was extended from symplectic to Spin^c -manifolds by Paradan and Vergne [25, 26, 27]. Their result for compact manifolds was generalised to

noncompact ones by the authors of this paper [17]. The latter result generalises to discrete series representations.

Let \mathcal{F} be the set of relative interiors of faces of the positive Weyl chamber \mathfrak{t}_+^* . For $\sigma \in \mathcal{F}$, let \mathfrak{k}_σ be the infinitesimal stabiliser of a point in σ . Let R_σ be the set of roots of $((\mathfrak{k}_\sigma)_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, and let $R_\sigma^+ := R_\sigma \cap R^+$. Set

$$\rho_\sigma := \frac{1}{2} \sum_{\alpha \in R_\sigma^+} \alpha.$$

Note that if σ is the interior of \mathfrak{t}_+^* , then $\rho_\sigma = 0$.

For any subalgebra $\mathfrak{h} \subset \mathfrak{k}$, let (\mathfrak{h}) be its conjugacy class. Set

$$\mathcal{H}_\mathfrak{k} := \{(\mathfrak{k}_\xi); \xi \in \mathfrak{k}\}.$$

For $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$, write

$$\mathcal{F}(\mathfrak{h}) := \{\sigma \in \mathcal{F}; (\mathfrak{k}_\sigma) = (\mathfrak{h})\}.$$

Let (\mathfrak{k}^M) be the conjugacy class (with respect to K) of the generic (i.e. minimal) infinitesimal stabiliser \mathfrak{k}^M of the action by K on M .

For $i\xi \in i\mathfrak{k}^*$, consider the *reduced space*

$$M_{i\xi} := \mu^{-1}(\xi)/G_\xi.$$

Here G_ξ is the stabiliser of ξ with respect to the coadjoint action. (Recall that we embed \mathfrak{k}^* into \mathfrak{g}^* as the annihilator of \mathfrak{p} .) By Propositions 3.13 and 3.14 in [14], we have $M_\xi = N_\xi$, including Spin^c -structures where relevant. The Spin^c -quantisation $Q^{\text{Spin}^c}(M_\xi) = Q^{\text{Spin}^c}(N_\xi)$ of such a reduced space, for the values ξ of μ we will need, is defined in Section 5.3 of [26].

Suppose the map μ is *G-proper*, in the sense that the inverse image of any cocompact set is cocompact.

Theorem 2.11 ($[Q^{\text{Spin}^c}, R] = 0$ for the discrete series). *In the setting of Proposition 2.8, we have*

$$\text{index}_{\text{ds}}(D_{\text{fv}}) = \bigoplus_{\lambda \in (\Lambda_+ + \rho_K)^{\text{reg}}} m_\lambda \pi_\lambda^{\text{ds}},$$

where $(\Lambda_+ + \rho_K)^{\text{reg}} \subset \Lambda_+ + \rho_K$ is the subset of regular elements, and with $m_\lambda \in \mathbb{Z}$ given by

$$m_\lambda = \sum_{\substack{\sigma \in \mathcal{F}(\mathfrak{h}) \text{ s.t.} \\ \lambda - \rho_\sigma \in \sigma}} Q^{\text{Spin}^c}(M_{\lambda - \rho_\sigma}), \quad (2.12)$$

where $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$ is such that $([\mathfrak{k}^M, \mathfrak{k}^M]) = ([\mathfrak{h}, \mathfrak{h}])$. If no such \mathfrak{h} exists, then $\text{index}_{\text{ds}}(D_{\text{fv}}) = 0$, i.e. $m_\lambda = 0$ for all λ .

Remark 2.12. In [17], where compact groups are considered, it was not assumed that the set of zeroes of v is compact, only that μ is proper. The arguments in this paper actually show that Theorem 2.11 holds without Assumption 2.3. We have not included this generalisation here, because the definition of the index is less straightforward in that case.

2.8 Invariant Spin^c -quantisation commutes with reduction

Now let G be any unimodular Lie group, such that G/K is even-dimensional, and G/K is equivariantly Spin . In this subsection we suppose in addition that G is reductive. Consider any complete, G -invariant Riemannian metric on TM . In Theorem 6.8 in [14], the invariant index of deformed Spin^c -Dirac operators was shown to satisfy an asymptotic version of the quantisation commutes with reduction principle. This result can be sharpened. Consider the multiplicities $n_\lambda \in \mathbb{Z}$ in

$$\pi_p = \sum_{\lambda \in \Lambda_+ + \rho_K} n_\lambda \pi_\lambda^K.$$

Theorem 2.13 ($[Q^{\text{Spin}^c}, R] = 0$ for the trivial representation). *If f is ψ -admissible, then one has for any G -invariant, complete Riemannian metric on M ,*

$$\text{index}_{L^2_T}^G(D_{fv}) = \sum_{\lambda \in \Lambda_+ + \rho_K} n_\lambda m_\lambda,$$

with m_λ as in (2.12).

In the cocompact case, every smooth section is transversally L^2 . Hence

$$\text{index}_{L^2_T}^G(D) = \dim(\ker(D^+)^G) - \dim(\ker(D^-)^G). \quad (2.13)$$

(See Theorem 2.7 in [24].) By Lemma D.2 and Proposition D.3 in Bunke's appendix to [24], the above integer equals

$$I_*^G(\mu_M^G[D]), \quad (2.14)$$

where $I^G: C^*G \rightarrow \mathbb{C}$ is defined by integrating functions over G , on the dense subalgebra $L^1(G) \subset C^*G$. Now we use the maximal group C^* -algebra rather than the reduced one. Furthermore, Assumption 2.3 holds automatically now. We do not need to assume that the Riemannian metric is a product metric, because the K -homology class of D does not depend on the Riemannian metric. Therefore, we obtain the following Spin^c -version

of Landsman's quantisation commutes with reduction conjecture [16, 22]. Compared to the main result in [24], this result applies in the more general Spin^c -setting, and also holds exactly, rather than asymptotically. (The assumptions that G is reductive and unimodular, and G/K is equivariantly Spin are not made in [24], however.)

Corollary 2.14 (Spin^c -Landsman conjecture). *If M/G is compact, then for any G -invariant Riemannian metric on M , we have*

$$I_*^G(\mu_M^G[D]) = \sum_{\lambda \in \Lambda_+ + \rho_K} n_\lambda m_\lambda,$$

with m_λ as in (2.12).

2.9 Vanishing results on Spin-manifolds

In [3], Atiyah and Hirzebruch showed that for compact, connected Lie groups (or equivalently, for circles) the equivariant index of a Spin-Dirac operator on a compact Spin manifold is zero for nontrivial actions. This was generalised to cocompact actions in a K -theoretical setting in [15]. There are also versions for the discrete series index and the invariant index.

The action by G on M is called *properly trivial* if every stabiliser group is maximal compact (i.e., is as large as it can be). Otherwise it is called properly nontrivial. Let G be as in the previous subsection, but without assuming it to be reductive.

Theorem 2.15. *Suppose M is G -equivariantly Spin, and that the action is cocompact and properly nontrivial. If D is the Spin-Dirac operator, then*

- *For any G -invariant Riemannian metric on M , $\text{index}_{L_1^2}^G(D) = 0$.*
- *If G is connected and semisimple with discrete series, and the Riemannian metric on TM is a product metric, then $\text{index}_{ds}(D) = 0$.*

In the cocompact case, we saw that $\text{index}_{L_1^2}^G(D)$ equals (2.13) and (2.14). Because of the latter equality, the first part of Theorem 2.15 also follows from the result in [15].

3 Decomposing the Dirac operator

The proofs of the results in Section 2 are based on two induction results: Propositions 4.5 and 4.11. We prove these by decomposing the Spin^c Dirac operator D in an explicit way, as discussed in this section.

In this section, unless stated otherwise, G is any Lie group for which the adjoint action by K on \mathfrak{p} lifts to $\text{Spin}(\mathfrak{p})$. As before, suppose that $K < G$ is a maximal compact subgroup, and that M and G/K are even-dimensional. (Unimodularity of G is not used in this section.) We assume that the Riemannian metric on M is a product metric.

3.1 Dirac operators on N and G/K

Let $P \rightarrow M$ be the G -equivariant Spin^c -structure used before. In Section 3.2 of [11] and Section 3.2 of [14], an induction procedure of equivariant Spin^c -structures from N to M is described. Proposition 3.10 of [14] is a Spin^c -slice theorem, which states that there is a K -equivariant Spin^c -structure P_N on N , such that the induced Spin^c -structure on M equals the Spin^c -structure originally given. The connection ∇^L on $L \rightarrow M$ restricts to a connection on the determinant line bundle $L_N = L|_N$ of this Spin^c -structure on N . This defines a Spin^c -momentum map $\mu_N: N \rightarrow \mathfrak{k}^*$, analogously to (2.4). In Lemma 5.3 of [14], it was shown that the connection ∇^L can be chosen such that $\mu(N) \subset \mathfrak{k}^*$, and

$$\mu_N = \mu|_N. \quad (3.1)$$

Since M and G/K are even-dimensional, so is N . Let $\mathcal{S}_N \rightarrow N$ be the spinor bundle associated to P_N . Let $\nabla^{\mathcal{S}_N}$ be the spinor connection on \mathcal{S}_N defined by the Levi-Civita connection on TN and the connection $\nabla^L|_N$ on $L_N = L|_N$. Let D_N be the associated Spin^c -Dirac operator on \mathcal{S}_N .

By Lemma 6.2 in [11], one has an equivariant vector bundle isomorphism

$$\mathcal{S} \cong G \times_K (\mathcal{S}_N \otimes \pi_p). \quad (3.2)$$

(Here we use graded tensor products.) At the level of smooth sections, we get

$$\Gamma^\infty(\mathcal{S}) \cong (\Gamma^\infty(G \times N, p_N^* \mathcal{S}_N) \otimes \pi_p)^K, \quad (3.3)$$

where $p_N: G \times N \rightarrow N$ is the natural projection map.

For $s \in \Gamma^\infty(\mathcal{S}_N)$ and $\varphi \in C^\infty(G) \otimes \pi_p$, define $\sigma(s \otimes \varphi) \in \Gamma^\infty(p_N^* \mathcal{S}) \otimes \pi_p$ by

$$(\sigma(s \otimes \varphi))(g, n) = s(n) \otimes \varphi(g),$$

for $n \in N$ and $g \in G$. Let ε be the grading operator on \mathcal{S}_N , equal to ± 1 on \mathcal{S}_N^\pm .

Proposition 3.1. *The map σ , together with (3.3), defines a G -equivariant linear isomorphism*

$$(\Gamma^\infty(\mathcal{S}_N) \hat{\otimes} C^\infty(G) \otimes \pi_p)^K \cong \Gamma^\infty(\mathcal{S}),$$

where $\hat{\otimes}$ denotes the tensor product completed in the Fréchet topology on $\Gamma^\infty(\mathcal{S})$. Under this isomorphism, the Dirac operator D corresponds to

$$D_N \otimes 1 + \varepsilon \otimes D_{G/K}, \quad (3.4)$$

where $D_{G/K}$ was defined in (2.3).

Remark 3.2. If N is a point, then Proposition 3.1 reduces to Proposition 1.1 in [28] (where one takes V to be the trivial representation). If $G = K$ is compact, then one gets the trivial identity $D_N = D_N$. In Proposition 6.7 of [11], it was shown that Proposition 3.3 holds at the level of principal symbols.

3.2 A reformulation

We will in fact first prove a reformulation of Proposition 3.1, and then deduce this proposition.

With respect to the decompositions (2.2) and (3.2), the Clifford action c by TM on \mathcal{S} is given by

$$c[g, v, X][g, s_N, y] = [g, c_N(v)s_N, y] + [g, \varepsilon s_N, c_p(X)y]. \quad (3.5)$$

Here $g \in G$, $n \in N$, $v \in T_n N$, $X \in \mathfrak{p}$, $s_N \in (\mathcal{S}_N)_n$, $y \in \pi_p$, and we used the Clifford actions $c_N: TN \rightarrow \text{End}(\mathcal{S}_N)$ and $c_p: \mathfrak{p} \rightarrow \text{End}(\pi_p)$. Let $p_N^* D_N$ be the operator on $\Gamma^\infty(p_N^* \mathcal{S}_N)$ given by

$$(p_N^* D_N s)(g, n) = D_N(s(g, -))(n),$$

for $s \in \Gamma^\infty(p_N^* \mathcal{S}_N)$, $g \in G$ and $n \in N$.

Fix an orthonormal basis $\{X_1, \dots, X_k\}$ of \mathfrak{p} , and consider the operator

$$D_p := \sum_{j=1}^k \mathcal{L}_{X_j} \otimes c_p(X_j) \quad (3.6)$$

on $\Gamma^\infty(\mathcal{S})$, via (3.3), where \mathcal{L}_{X_j} is the Lie derivative of sections of $p_N^* \mathcal{S}_N$ with respect to X_j . Then we have the following decomposition of D .

Proposition 3.3. *Under the identification (3.3), one has*

$$D = p_N^* D_N + \varepsilon D_p,$$

restricted to K -invariant sections.

This result implies Proposition 3.1.

Proof of Proposition 3.1. The map σ maps K -invariant sections to K -invariant sections, and its image is dense in $(\Gamma^\infty(p_N^* \mathcal{S}) \otimes \pi_p)^K$. Furthermore, with notation as above,

$$\begin{aligned} (\sigma(D_N s \otimes \varphi + \varepsilon s \otimes D_{G/K} \varphi))(g, n) &= (D_N s)(n) \otimes \varphi(g) + \varepsilon s(n) \otimes (D_{G/K} \varphi)(g) \\ &= ((p_N^* D_N + \varepsilon D_p) \sigma(s \otimes \varphi))(g, n). \end{aligned}$$

Proposition 3.3 states that $p_N^* D_N + \varepsilon D_p$, restricted to K -invariant sections, is the Dirac operator D . \square

It remains to prove Proposition 3.3.

3.3 The Levi–Civita connection

To prove Proposition 3.3, we start by decomposing the Levi–Civita connection on TM . Let ∇^N be the Levi–Civita connection on TN , and let $\nabla^{G/K}$ be the Levi–Civita connection on $T(G/K)$, for the Riemannian metric defined by the given inner product on \mathfrak{p} . Consider the projection map $p_{G/K}: G \times N \rightarrow G/K$. Using

$$p_{G/K}^* T(G/K) = p_{G/K}^* (G \times_K \mathfrak{p}) = G \times N \times \mathfrak{p} \rightarrow G \times N,$$

we rewrite (2.2) as

$$TM = (p_N^* TN \oplus p_{G/K}^* T(G/K))/K.$$

In terms of the action map $p_M: G \times N \rightarrow M$, this can be rephrased as

$$p_M^* TM = p_N^* TN \oplus p_{G/K}^* T(G/K).$$

We find that the space $\mathfrak{X}(M)$ of vector fields on M decomposes as

$$\mathfrak{X}(M) = \Gamma^\infty(G \times N, p_N^* TN \oplus p_{G/K}^* T(G/K))^K.$$

Consider the connection

$$\tilde{\nabla}^M := p_N^* \nabla^N \oplus p_{G/K}^* \nabla^{G/K}$$

on $p_N^* TN \oplus p_{G/K}^* T(G/K)$. Let ∇^M be the connection on TM equal to the restriction of $\tilde{\nabla}^M$ to K -invariant sections. In other words, $p_M^* \nabla^M = \tilde{\nabla}^M$.

Lemma 3.4. *The connection ∇^M is the Levi–Civita connection on TM .*

Proof. The fact that ∇^N and $\nabla^{G/K}$ preserve the Riemannian metrics on N and G/K , respectively, implies that ∇^M preserves the Riemannian metric on M . Here we use the fact that the Riemannian metric on TM is a product metric.

To show that ∇^M is torsion-free, we note that the torsion Tor^{∇^M} of ∇^M is a tensor, so it is enough to show it vanishes on a set of vector fields spanning TM . Therefore, we only need to show it vanishes on $(K$ -invariant) vector fields of the forms $p_N^* v_N$ and $p_{G/K}^* v_{G/K}$, for $v_N \in \mathfrak{X}(N)$ and $v_{G/K} \in \mathfrak{X}(G/K) \cong ((C^\infty(G) \otimes \mathfrak{p}))^K$.

Now for $v_N, w_N \in \mathfrak{X}(N)$, we have

$$\nabla_{p_N^* v_N}^M (p_N^* w_N) = (p_N^* \nabla^N)_{p_N^* v_N} (p_N^* w_N) = p_N^* (\nabla_{v_N}^N w_N).$$

Hence, because ∇^N is torsion-free,

$$\begin{aligned} \nabla_{p_N^* v_N}^M (p_N^* w_N) - \nabla_{p_N^* w_N}^M (p_N^* v_N) &= p_N^* (\nabla_{v_N}^N w_N - \nabla_{w_N}^N v_N) \\ &= p_N^* [v_N, w_N] \\ &= [p_N^* v_N, p_N^* w_N]. \end{aligned}$$

So

$$\text{Tor}^{\nabla^M} (p_N^* v_N, p_N^* w_N) = 0.$$

One similarly shows that for all $v_{G/K}, w_{G/K} \in \mathfrak{X}(G/K)$,

$$\text{Tor}^{\nabla^M} (p_{G/K}^* v_{G/K}, p_{G/K}^* w_{G/K}) = 0.$$

It therefore remains to show that

$$\text{Tor}^{\nabla^M} (p_N^* v_N, p_{G/K}^* v_{G/K}) = 0, \quad (3.7)$$

with v_N and $v_{G/K}$ as above.

Since each of the vector fields $p_N^* v_N$ and $p_{G/K}^* v_{G/K}$ is tangent to the directions the other vector field is constant in, their Lie bracket vanishes. Also,

$$\nabla_{p_N^* v_N}^M (p_{G/K}^* v_{G/K}) = (p_{G/K}^* \nabla^{G/K})_{p_N^* v_N} (p_{G/K}^* v_{G/K}) = 0,$$

since the tangent map of $p_{G/K}$ is zero on the image of $p_N^* v_N$. Similarly, one has $\nabla_{p_{G/K}^* v_{G/K}}^M (p_N^* v_N) = 0$. So in particular,¹

$$\nabla_{p_N^* v_N}^M (p_{G/K}^* v_{G/K}) - \nabla_{p_{G/K}^* v_{G/K}}^M (p_N^* v_N) = 0. \quad (3.8)$$

We conclude that (3.7) holds. So ∇^M is torsion-free, and hence indeed the Levi-Civita connection on TM . \square

¹Note that the Lie bracket of sections of $p_N^* TN$ and $p_{G/K}^* T(G/K)$, and the analogous expression to (3.8), do *not* vanish in general. This is only the case for the pulled-back sections considered here, which is enough.

3.4 Spinor connections

The decomposition of the Levi–Civita connection in Lemma 3.4 implies an analogous decomposition of the spinor connection $\nabla^{\mathcal{S}}$ on \mathcal{S} , associated to the connection ∇^L on $L \rightarrow M$.

Lemma 3.5. *In terms of the decomposition (3.3), one has for all $X \in \mathfrak{p}$ and $v \in \mathfrak{X}(N)$,*

$$\nabla_{p_N^*v+X^M}^{\mathcal{S}} = (p_N^*\nabla_N^{\mathcal{S}})_{p_N^*v} + \mathcal{L}_X, \quad (3.9)$$

where \mathcal{L}_X is the Lie derivative of sections of \mathcal{S} with respect to X .

Proof. Let $U \subset N$ be a K -invariant open subset such that

$$\mathcal{S}_N|_U = \mathcal{S}_0^U \otimes (L_N|_U)^{1/2},$$

where $\mathcal{S}_0^U \rightarrow U$ is the spinor bundle for a local Spin-structure on U . Then

$$\mathcal{S}|_{G \times_K U} = (G \times_K (\mathcal{S}_0^U \otimes \pi_p)) \otimes (G \times_K (L_N|_U))^{1/2}.$$

We have

$$\nabla^{\mathcal{S}}|_{G \times_K U} = \nabla^{\mathcal{S}_0^{G \times_K U}} \otimes 1 + 1 \otimes \nabla^{(L|_{G \times_K U})^{1/2}},$$

where $\nabla^{\mathcal{S}_0^{G \times_K U}}$ is the connection on the spinor bundle $\mathcal{S}_0^{G \times_K U} \rightarrow G \times_K U$ induced by the Levi–Civita connection on $G \times_K U \hookrightarrow M$.

First note that for all $s_{L_N} \in \Gamma^\infty(L_N)^K$, we have $p_N^*s_{L_N} \in \Gamma^\infty(p_N^*L_N)^K = \Gamma^\infty(L)$, and for all $X \in \mathfrak{p}$ and $v \in \mathfrak{X}(N)$,

$$\nabla_{p_N^*v+X^M}^L(p_N^*s_{L_N}) = p_N^*(\nabla_v^{L_N}s_{L_N}) = (p_N^*\nabla_N^{L_N})_{p_N^*v}p_N^*s_{L_N}.$$

This follows from the definition of ∇^L in (22) in [11]. Furthermore, let $\nabla^{\mathcal{S}_0^U}$ be the connection on \mathcal{S}_0^U induced by ∇^N , and let $\nabla^{\mathcal{S}_0^{G/K}}$ be the connection on the spinor bundle $\mathcal{S}_0^{G/K} = G \times_K \pi_p \rightarrow G/K$ induced by $\nabla^{G/K}$. Then Lemma 3.4 implies that one has

$$\nabla^{\mathcal{S}_0^{G \times_K U}} = p_N^*\nabla^{\mathcal{S}_0^U} + p_{G/K}^*\nabla^{\mathcal{S}_0^{G/K}},$$

restricted to K -invariant sections. The connection $\nabla^{\mathcal{S}_0^{G/K}}$ on $\Gamma^\infty(\mathcal{S}_0^{G/K}) = (C^\infty(G) \otimes \pi_p)^K$ is simply given by

$$\nabla_{X^{G/K}}^{\mathcal{S}_0^{G/K}} s_{G/K} = \mathcal{L}_X(s_{G/K}),$$

for $X \in \mathfrak{p}$ and $s_{G/K} \in (C^\infty(G) \otimes \pi_{\mathfrak{p}})^K$. (As noted on page 7 of [28], the connection $\nabla^{G/K}$ is induced by the canonical connection on the principal fibre bundle $G \rightarrow G/K$.)

Since both sides of (3.9) satisfy the Leibniz rule, it is enough to check this equality on a set of sections spanning $\mathcal{S}|_{\mathcal{U}}$. Hence it is enough to consider a section

$$s := p_N^*(s_N \otimes s_{L_N}) \otimes p_{G/K}^* s_{G/K} \in \Gamma^\infty(\mathcal{S}|_{G \times_K \mathcal{U}}),$$

for

$$\begin{aligned} s_N &\in \Gamma^\infty(\mathcal{S}_0^{\mathcal{U}})^K; \\ s_{G/K} &\in (C^\infty(G) \otimes \pi_{\mathfrak{p}})^K; \\ s_{L_N} &\in \Gamma^\infty(L_N|_{\mathcal{U}}^{1/2})^K. \end{aligned}$$

For such a section, and for all $X \in \mathfrak{p}$ and $v \in T\mathcal{U}$, the preceding arguments allow us to compute

$$\begin{aligned} \nabla_{p_N^* v + X^M}^{\mathcal{S}} s &= \nabla_{p_N^* v + X^M}^{\mathcal{S}_0^{G \times_K \mathcal{U}}} (p_N^* s_N \otimes p_{G/K}^* s_{G/K}) \otimes p_N^* s_{L_N} \\ &\quad + p_N^* s_N \otimes p_{G/K}^* s_{G/K} \otimes \nabla_{p_N^* v + X^M}^{L|_{G \times_K \mathcal{U}}^{1/2}} p_N^* s_{L_N} \\ &= (p_N^* \nabla^{\mathcal{S}_0^{\mathcal{U}}})_{p_N^* v} (p_N^* s_N) \otimes p_{G/K}^* s_{G/K} \otimes p_N^* s_{L_N} \\ &\quad + p_N^* s_N \otimes \mathcal{L}_X(p_{G/K}^* s_{G/K}) \otimes p_N^* s_{L_N} \\ &\quad + p_N^* s_N \otimes p_{G/K}^* s_{G/K} \otimes p_N^* (\nabla_v^{L_N|_{\mathcal{U}}^{1/2}} s_{L_N}) \\ &= (p_N^* \nabla^{\mathcal{S}_N})_{p_N^* v} (p_N^* (s_N \otimes s_{L_N})) \otimes p_{G/K}^* s_{G/K} \\ &\quad + p_N^* (s_N \otimes s_{L_N}) \otimes \mathcal{L}_X(p_{G/K}^* s_{G/K}) \\ &= ((p_N^* \nabla^{\mathcal{S}_N})_{p_N^* v} + \mathcal{L}_X) s, \end{aligned}$$

since $(p_N^* \nabla^{\mathcal{S}_N})_{p_N^* v}$ vanishes on sections pulled back from G/K , while X vanishes on sections pulled back from N . \square

3.5 Proof of Proposition 3.3

Using Lemma 3.5, we can prove Proposition 3.3. One ingredient of the proof is the following expression for the operator $p_N^* D_N$.

Lemma 3.6. *If $\{e_1, \dots, e_l\}$ is a local orthonormal frame for TN , then locally,*

$$p_N^* D_N = \sum_{s=1}^l c(p_N^* e_s) (p_N^* \nabla^{\mathcal{S}_N})_{p_N^* e_s}. \quad (3.10)$$

Proof. Note that any section of $\Gamma^\infty(p_N^* \mathcal{S}_N)$ is a sum of sections of the form $\varphi p_N^* s_N$, for $\varphi \in C^\infty(G \times N)$ and $s_N \in \Gamma^\infty(\mathcal{S}_N)$. On such a section, one has

$$(p_N^* \nabla^{\mathcal{S}_N})_{p_N^* e_s} (\varphi p_N^* s_N) = \varphi p_N^* (\nabla_{p_N^* e_s}^{\mathcal{S}_N} s_N) + (p_N^* e_s)(\varphi) p_N^* s_N. \quad (3.11)$$

At a point $(g, n) \in G \times N$, one has

$$(p_N^* e_s)(\varphi)(g, n) = e_s(\varphi(g, -))(n).$$

Therefore, at such a point, we find that (3.11) equals

$$\left(\nabla_{e_s}^{\mathcal{S}_N} \varphi(g, -) s_N \right) (n).$$

We conclude that, at (g, n) , the right hand side of (3.10) applied to $\varphi p_N^* s_N$ yields

$$\begin{aligned} \left(\sum_{s=1}^l c(p_N^* e_s) \nabla_{p_N^* e_s}^{\mathcal{S}_N} (\varphi p_N^* s_N) \right) (g, n) &= \sum_{s=1}^l \left(c(e_s) \left(\nabla_{e_s}^{\mathcal{S}_N} \varphi(g, -) s_N \right) \right) (n) \\ &= ((p_N^* D_N)(\varphi p_N^* s_N))(g, n). \end{aligned}$$

□

Proof of Proposition 3.3. Let $\{X_1, \dots, X_k\}$ be an orthonormal basis of \mathfrak{p} , and let $\{e_1, \dots, e_l\}$ be a local orthonormal frame for TN . Then, because the Riemannian metric on TM is a product metric,

$$D = \sum_{r=1}^k c(X_r) \nabla_{X_r}^{\mathcal{S}} + \sum_{s=1}^l c(p_N^* e_s) \nabla_{p_N^* e_s}^{\mathcal{S}}. \quad (3.12)$$

Note that for each r and s , $c(X_r)$ acts on π_p , and $c(p_N^* e_s)$ acts on \mathcal{S}_N in $\mathcal{S} = G \times_K (\mathcal{S}_N \otimes \pi_p)$, via (3.5).

By Lemma 3.5 and (3.5), the first term on the right hand side of (3.12) equals

$$\sum_{r=1}^k c(X_r) \mathcal{L}_{X_r} = \varepsilon D_p.$$

The same lemma implies that the second term equals

$$\sum_{s=1}^l c(p_N^* e_s) (p_N^* \nabla^{\mathcal{S}_N})_{p_N^* e_s},$$

which by Lemma 3.6 equals $p_N^* D_N$. □

4 Induction

We prove two induction results, Propositions 4.5 and 4.11, by using Proposition 3.1 and keeping track of the L^2 -norms on the various spaces involved. These induction results are then used to prove the results in Section 2. To compare L^2 -norms, we use a relation between the Riemannian densities on M , N and G .

In this section, initially G can be any Lie group, with a fixed Haar measure dg and maximal compact subgroup K . We still assume that the Riemannian metric on M is a product metric.

4.1 Densities

Recall that by assumption, the Riemannian metric on $M = G \times_K N$ is induced by the given inner product on \mathfrak{p} and a K -invariant Riemannian metric on N . Let dm and dn be the densities on M and N defined by these Riemannian metrics. Let dk be the Haar measure on K giving K unit volume. We will prove and use the fact that dm equals the measure $d[g, n]$ on $G \times_K N$ induced by the product measure $dg \times dn$ on $G \times N$, via the equality

$$\int_{G \times N} \varphi(g, n) dg dn = \int_{G \times_K N} \int_K \varphi(k \cdot \tau[g, n]) dk d[g, n]$$

for any $\varphi \in C_c(G \times N)$ and any Borel section $\tau: G \times_K N \rightarrow G \times N$. (See e.g. [6], Chapter 7, Section 2, Proposition 4b.)

Lemma 4.1. *Under the diffeomorphism $G \times_K N = M$ defined by the action, and for a suitable scaling of the Haar measure dg , the measure $d[g, n]$ corresponds to dm .*

Proof. Consider the non-equivariant diffeomorphisms

$$\begin{aligned} \Psi_M: \mathfrak{p} \times N &\rightarrow M; \\ \Psi_{G \times N}: \mathfrak{p} \times K \times N &\rightarrow G \times N, \end{aligned}$$

defined by

$$\begin{aligned} \Psi_M(X, n) &= \exp(X)n; \\ \Psi_{G \times N}(X, k, n) &= (\exp(X)k^{-1}, kn), \end{aligned}$$

for $X \in \mathfrak{p}$, $n \in N$ and $k \in K$.

Let dX be the Riemannian density on \mathfrak{p} . Then, since Ψ_M is an isometry,

$$\Psi_M^* dm = dX \otimes dn. \quad (4.1)$$

Now let the Haar measure dg be given by the G -invariant Riemannian metric induced by the inner product on \mathfrak{g} . Let dk be the Haar measure on K defined in the same way. By rescaling the inner product on \mathfrak{g} , we can make sure that dk gives K unit volume. By Lemma 4.2 below, we have

$$\Psi_{G \times N}^*(dg \otimes dn) = dX \otimes dk \otimes dn. \quad (4.2)$$

The equalities (4.1) and (4.2) imply that for all $\varphi \in C_c(M)$,

$$\begin{aligned} \int_M \varphi(m) dm &= \int_{\mathfrak{p} \times N} \varphi(\exp(X)n) dX \otimes dn \\ &= \int_{\mathfrak{p} \times K \times N} \varphi(\exp(X)n) dX \otimes dk \otimes dn \\ &= \int_{G \times N} \varphi(gn) dg \otimes dn \\ &= \int_{G \times_K N} \varphi(gn) d[g, n], \end{aligned}$$

where we used the fact that the map $(g, n) \mapsto gn$ is invariant under the K -action given by $k \cdot (g, n) = (gk^{-1}, kn)$. \square

Lemma 4.2. *In the notation of the proof of Lemma 4.1, we have*

$$\Psi_{G \times N}^*(dg \otimes dn) = dX \otimes dk \otimes dn.$$

Proof. One can compute that for all $X, Y \in \mathfrak{p}$, $Z \in \mathfrak{k}$, $k \in K$, $n \in N$ and $v \in T_n N$,

$$T_{(X, k, n)} \Psi_{G \times N}(Y, T_e l_k(Z), v) = (T_e l_{\exp(X)k^{-1}}(\text{Ad}(k)(Y + Z)), T_n k(\alpha_n(Z) + v)).$$

Here the letter l denotes left multiplication, and for $m \in M$, the map $\alpha_m: \mathfrak{g} \rightarrow T_m M$ is given by the infinitesimal action. Now the maps $T_e l_{\exp(X)k^{-1}}$, $\text{Ad}(k)$ and $T_n k$ preserve the Riemannian metrics on TG and TN . So

$$T_{(X, k, n)} \Psi_{G \times N} = B \circ A,$$

where

$$A: T_{(X, k, n)}(\mathfrak{p} \times K \times N) \rightarrow T_{(\exp(X)k^{-1}, kn)}(G \times N),$$

given by

$$A(Y, T_e l_k(Z), v) = (T_e l_{\exp(X)k^{-1}}(\text{Ad}(k)(Y + Z)), T_n k(v))$$

is an isometry, and the automorphism B of

$$T_{(\exp(X)k^{-1}, kn)}(G \times N) \cong \mathfrak{p} \oplus \mathfrak{k} \oplus T_{kn}N$$

is given by the matrix

$$\text{mat}(B) = \begin{pmatrix} I_{\mathfrak{p}} & 0 & 0 \\ 0 & I_{\mathfrak{k}} & 0 \\ 0 & T_n k \circ \alpha_n & I_{T_{kn}N} \end{pmatrix},$$

where $I_{\mathfrak{k}}$, $I_{\mathfrak{p}}$ and $I_{T_{kn}N}$ are the identity maps on the respective spaces, so that B has determinant one.

Since the map A is an isometry, it relates the Riemannian density $dX \otimes dk \otimes dn$ on $\mathfrak{p} \times K \times N$ to the Riemannian density $dg \otimes dn$ on $G \times N$, at the point (X, k, n) . Since the map B has unit determinant, it does not change densities, so the claim follows. \square

Lemma 4.3. *In the notation of Proposition 3.1, we have*

$$\|\sigma(s \otimes \varphi)\|_{L^2(\mathcal{S})} = \|s\|_{L^2(\mathcal{S}_N)} \|\varphi\|_{L^2(G) \otimes \pi_{\mathfrak{p}}},$$

for all $s \in \Gamma_c^\infty(\mathcal{S}_N)$ and $\varphi \in C_c^\infty(G) \otimes \pi_{\mathfrak{p}}$ such that $s \otimes \varphi$ is K-invariant.

Proof. By Lemma 4.1 and K-invariance of $s \otimes \varphi$ and of the norm on \mathcal{S} , and implicitly using a Borel section $G \times_K N \rightarrow G \times N$, one has

$$\begin{aligned} \|\sigma(s \otimes \varphi)\|_{L^2(\mathcal{S})}^2 &= \int_{G \times_K N} \|s(n) \otimes \varphi(g)\|_{\mathcal{S}}^2 d[g, n] \\ &= \int_{G \times_K N} \int_K \|s(n) \otimes \varphi(g)\|_{\mathcal{S}}^2 dk d[g, n] \\ &= \int_{G \times N} \|s(n)\|_{\mathcal{S}_N}^2 \|\varphi(g)\|_{\pi_{\mathfrak{p}}}^2 dg dn \\ &= \|s\|_{L^2(\mathcal{S}_N)}^2 \|\varphi\|_{L^2(G) \otimes \pi_{\mathfrak{p}}}^2. \end{aligned}$$

\square

4.2 Deformed Dirac operators

Now suppose G/K is even-dimensional and equivariantly Spin. Consider a real-valued function $f \in C^\infty(M)^G = C^\infty(N)^K$, and the deformed Dirac operator

$$D_{fv_N} := D_N + ic_N(fv_N).$$

Proposition 4.4. *The map σ defines a G -equivariant, graded, unitary isomorphism*

$$\ker_{L^2} D_{fv} \cong (\ker_{L^2}(D_{fv_N}) \otimes \ker_{L^2}(D_{G/K}))^K$$

(Here the tensor product is completed in the L^2 -inner product.)

Proof. Since the algebraic tensor product

$$\Gamma_c^\infty(\mathcal{S}_N) \otimes C_c^\infty(G) \otimes \pi_p$$

is dense in

$$L^2(\mathcal{S}_N) \otimes L^2(G) \otimes \pi_p,$$

Proposition 3.1 and Lemma 4.3 imply that σ induces a unitary isomorphism

$$L^2(\mathcal{S}) \cong (L^2(\mathcal{S}_N) \otimes L^2(G) \otimes \pi_p)^K.$$

By Proposition 3.1, this isomorphism intertwines the operators D_M and $D_N \otimes 1 + \varepsilon \otimes D_{G/K}$. Since it also intertwines $c(v)$ and $c(v_N) \otimes 1$, it intertwines D_{fv} and $D_{fv_N} \otimes 1 + \varepsilon \otimes D_{G/K}$.

As in the proof of Theorem 3.5 in [2], the presence of the grading operator ε in (3.4) implies that

$$(D_{fv_N} \otimes 1 + \varepsilon \otimes D_{G/K})^2 = D_{fv_N}^2 \otimes 1 + 1 \otimes D_{G/K}^2.$$

Since the operators D_{fv_N} and $D_{G/K}$ are symmetric, we find that

$$\ker_{L^2} D_{fv} \cong (\ker_{L^2}(D_{fv_N}) \otimes \ker_{L^2}(D_{G/K}))^K.$$

Since the isomorphism is compatible with the gradings, the claim follows. \square

Proposition 4.4 holds at the level of kernels. To prove the results in Section 2, we only need the corresponding weaker result about indices. Suppose G is semisimple with discrete series. Consider the Dirac induction map (2.10). Note that for $(\lambda \in \Lambda_+ + \rho_K)^{\text{reg}}$, we have

$$\widehat{D\text{-Ind}}_K^G(\pi_\lambda^K) = \pi_\lambda^{\text{ds}}.$$

The following induction result for indices follows directly from Proposition 4.4 and Theorem 2.2.

Proposition 4.5. *In the setting of Proposition 2.8, we have*

$$\text{index}_{\text{ds}} D_{\text{fv}} = \widehat{\text{D-Ind}}_K^G(\text{index}_K D_{\text{fv}_N}).$$

4.3 Invariant parts

In this subsection, G is unimodular, and G/K is even-dimensional and equivariantly Spin. We now consider G -invariant, transversally L^2 sections of \mathcal{S} , to prove Proposition 4.11.

Lemma 4.6. *Restriction to N is a linear isomorphism*

$$\Gamma^\infty(\mathcal{S})^G \xrightarrow{\cong} (\Gamma^\infty(\mathcal{S}_N) \otimes \pi_p)^K.$$

Proof. Note that for all $s \in \Gamma^\infty(\mathcal{S})^G$ and $n \in N$, we have $s(n) \in \mathcal{S}_n \cong (\mathcal{S}_N)_n \otimes \pi_p$. Every K -invariant section in $(\Gamma^\infty(\mathcal{S}_N) \otimes \pi_p)^K$ has a unique G -invariant extension to a section in $\Gamma^\infty(\mathcal{S})^G$. This is the inverse to the restriction map. \square

Fix $s \in \Gamma^\infty(\mathcal{S})^G$. Let $h_G \in C^\infty(G)^K$ be such that

$$\int_G h_G(g)^2 dg = 1$$

for the Haar measure dg as in Lemma 4.1. Here the superscript K denotes invariance under right multiplication by K . Define the cutoff function $h \in C^\infty(M)$ by

$$h(gn) = h_G(g),$$

for $g \in G$ and $n \in N$.

The characterisation of the density dm in Lemma 4.1 allows us to relate transversally L^2 sections on M to L^2 -sections on N .

Lemma 4.7. *We have*

$$\|hs\|_{L^2(\mathcal{S})} = \|s|_N\|_{L^2(\mathcal{S}_N) \otimes \pi_p}.$$

Proof. By Lemma 4.1, we have

$$\begin{aligned}
\|hs\|_{L^2(\mathcal{S})}^2 &= \int_{\mathcal{M}} h(m)^2 \|s(m)\|_{\mathcal{S}}^2 dm \\
&= \int_{G \times_K N} h_G(g)^2 \|g^{-1}(s(n))\|_{\mathcal{S}}^2 d[g, n] \\
&= \int_{G \times N} h_G(g)^2 \|s(n)\|_{\mathcal{S}_N \otimes \pi_p}^2 dg dn \\
&= \|s|_N\|_{L^2(\mathcal{S}_N) \otimes \pi_p}^2,
\end{aligned}$$

where we have used G -invariance of the metric $\|\cdot\|_{\mathcal{S}}$ and K -invariance of $s|_N$. \square

Unimodularity of G implies that the definition of the space $L_T^2(\mathcal{S})^G$ is independent of the cutoff function chosen. Lemma 4.7 has the following consequence.

Lemma 4.8. *Restriction to N is a graded unitary isomorphism*

$$L_T^2(\mathcal{S})^G \cong (L^2(\mathcal{S}_N) \otimes \pi_p)^K.$$

In Proposition 3.1, the operator $D_{G/K}$ is zero on G -invariant sections. It therefore has the following consequence.

Lemma 4.9. *One has*

$$(Ds)|_N = (D_N \otimes 1_{\pi_p})(s|_N).$$

Because of (3.1), we have $v_N = v|_N$. Therefore,

$$(c(v)s)|_N = c_N(v_N)s|_N \tag{4.3}$$

Lemmas 4.8 and 4.9, together with (4.3), yield the following conclusion.

Proposition 4.10. *We have a graded linear isomorphism*

$$\ker_{L_T^2}^G(D_{fv}) \cong (\ker_{L^2}(D_{fv_N}) \otimes \pi_p)^K.$$

Using Proposition 4.10 and the fact that $\pi_p \cong \pi_p^*$, we obtain the desired induction result.

Proposition 4.11. *We have*

$$\text{index}_{L_T^2}^G(D_{fv}) = [\text{index}_K(D_{fv_N}) : \pi_p] \in \mathbb{Z}.$$

We have so far assumed that the Riemannian metric on M is a product metric in this section. However, the invariant index is independent of the (complete, G -invariant) Riemannian metric by cobordism invariance, Theorem 3.6 in [8]. Furthermore, any Riemannian manifold with a complete, G -invariant Riemannian metric has a complete product metric by Lemma 3.12 in [19]. Therefore, Proposition 4.11 holds for *any* complete, G -invariant Riemannian metric on TM .

4.4 Proofs of the results

Let us prove the results in Section 2. Proposition 2.8 follows directly from Theorem 2.2, Proposition 4.4 and well-definedness of the index (2.8) for compact groups.

Corollary 3.8 in [19] implies that

$$\text{index}_G(D_{fv}) = \text{index}_K(D_{fv_N}) \otimes \pi_p \in \hat{R}(K). \quad (4.4)$$

Hence Proposition 2.4 follows from Proposition 4.11:

$$\text{index}_{L^2}^G(D_{fv}) = \dim(\text{index}_K D_{fv_N} \otimes \pi_p)^K = \dim(\text{index}_G(D_{fv})^K).$$

In the same way, Proposition 2.9 follows from (4.4) and Proposition 4.5.

To prove Proposition 2.10, we note that by (5.3) in [12],

$$D\text{-Ind}_K^G(\pi_\lambda^K) = (-1)^{\dim(G/K)/2} j(\pi_\lambda^{\text{ds}}) \in K_0(C_r^*G).$$

Here $D\text{-Ind}_K^G$ is the K -theoretical Dirac induction map from the Connes–Kasparov conjecture (see Conjecture 4.20 in [5]). Therefore, by the induction result for the analytic assembly map, Theorem 5.8 in [14], we have

$$\begin{aligned} p_{\text{ds}}(\mu_M^G[D]) &= p_{\text{ds}} \circ D\text{-Ind}_K^G(\text{index}_K D_N) \\ &= (-1)^{\dim(G/K)/2} \bigoplus_{\lambda \in (\Lambda_+ + \rho_K)^{\text{reg}}} [\text{index}_K(D) : \pi_\lambda^K] j(\pi_\lambda^{\text{ds}}) \\ &= (-1)^{\dim(G/K)/2} j(\widehat{D\text{-Ind}_K^G}(\text{index}_K(D_N))) \\ &= (-1)^{\dim(G/K)/2} j(\text{index}_{\text{ds}}(D)), \end{aligned}$$

by Proposition 4.5.

Theorem 2.11 follows from the corresponding result for compact groups, Theorem 3.10 in [17], via Proposition 4.5. Here we also use the fact that $M_\xi = N_\xi$ for all $\xi \in \mathfrak{k}^*$, if G is reductive (see Proposition 3.13 in [14]). In

a similar way, we can use Proposition 4.11 to deduce Theorem 2.13 from Theorem 3.9 in [17]. In that case, we do not need to assume that the Riemannian metric on TM is a product metric.

Finally, Theorem 2.15 follows from Atiyah and Hirzebruch's result in [3] via Propositions 4.5 and 4.11. Indeed, the condition that the action is properly nontrivial is equivalent to the action by K on N being nontrivial, see Lemma 9 in [15]. Also, the Spin^c -structure P_N is now a Spin-structure, see Lemma 10 in [15].

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